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Supersymmetric completion of M-theory 4D-gauge algebra from twisted tori and fluxes

Riccardo D'Auria,[a] Sergio Ferrara,[b] and Mario Trigiante[a]

[a] *Dipartimento di Fisica Politecnico di Torino*
C.so Duca degli Abruzzi, 24, I-10129 Torino, and
Istituto Nazionale di Fisica Nucleare
Sezione di Torino, Italy
[b] *CERN, Physics Department*
CH 1211 Geneva 23, Switzerland, and
INFN, Laboratori Nazionali di Frascati
Italy, and
Department of Physics & Astronomy, University of California
Los Angeles, CA, U.S.A.
E-mail: riccardo.dauria@polito.it, sergio.ferrara@cern.ch, mario.trigiante@polito.it

Abstract: We present the supersymmetric completion of the M-theory free differential algebra resulting from a compactification to four dimensions on a twisted seven-torus with 4-form and 7-form fluxes turned on. The super-curvatures are given and the local supersymmetry transformations derived. Dual formulations of the theory are discussed in connection with classes of gaugings corresponding to diverse choices of vacua. This also includes seven dimensional compactifications on more general spaces not described by group manifolds.

Keywords: Supergravity Models, M-Theory, Flux compactifications.

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Contents

1. Introduction

Recently superstring and M-theory compactifications on manifolds admitting globally defined spinors [1−9] (for a recent review on string/M-theory flux-compactifications see also [10]), but with broken supersymmetry, have renewed considerable attention in as much as they offer examples of theories with a low-energy effective lagrangian exhibiting spontaneously broken local supersymmetry as well as Higgs phases of certain gauge symmetries.

A popular example of this class of theories corresponds to the compactification on twisted tori with form-fluxes turned on [11−18]. These models [19−26], depending on the choice of fluxes, can give rise to no-scale supergravities [27] with partially broken supersymmetry or other type of vacua. These solutions can have either Minkowski or AdS geometry. In the case of M-theory it turns out that the 4-form flux on the internal space has to be trivial. The solutions with AdS geometry are characterized by non trivial 7-form flux and define eleven dimensional backgrounds of Freund-Rubin type [28] with flat directions. This has to be contrasted with the maximal $N = 8$ model when the internal space in S^7 and no flat directions remain [29].

Twisted tori can be described as seven dimensional group manifolds whose isometries are part of the gauge group of the theory [11]. This seven dimensional gauge algebra, which is always spontaneously broken for flat groups [11], enlarges to a bigger symmetry when the vectors (or their dual) coming from the 3-form are also included, thus realizing a non-trivial 28-dimensional subalgebra [30] of the maximal rigid symmetry $\mathrm{E}_{7(7)}$ [31]. In this context we show how a suitable choice of fluxes of M-theory, different from the torus twist, gives rise to the $N = 8$ gauged SO(8) supergravity [29], which is known to exhibit an AdS vacuum with maximal unbroken supersymmetry. Other choices, which still have a 7-dimensional interpretation, are also possible and some examples are presented here.

The main issue discussed in this article (sections 2 and 3) is the supersymmetric completion of the set of curvatures and gauge transformations which modify the free differential algebra (FDA) of M-theory compactified to four dimensions with fluxes [32 − 34, 30, 35, 36]. As a result we also give the spin 3/2 and spin 1/2 curvatures as well as the gauge transformation laws of the fields and the local supersymmetry transformations. These results allow to give the full structure of the super-FDA, with the curvatures turned on, as it is in general the case if we consider configurations other than the vacuum, which, in many cases, would not even exist. This is the case for instance of massive type-IIA supergravity which does not admit a (zero curvature) vacuum solution.

In section 4 we discuss different classes of gaugings, using the embedding tensor method, introduced and developed in [37 − 41], and retrieve some examples of M-theory vacua discussed in the literature, which do not necessary fall on group manifold compactifications.

2. Curvatures of the supersymmetric FDA

In this section we give the supersymmetric FDA in four dimensions as obtained by dimensional reduction of eleven dimensional supergravity [42] on a twisted seven-torus with 4-form flux turned on [35].

We shall denote by $r, s, t = 0, \ldots, 3$ and by $a, b, c = 4, \ldots, 10$ the rigid four and seven dimensional indices, and by μ, $\nu = 0, \ldots, 3$ and $I, J, M, N = 4, \ldots, 10$ the curved ones respectively. The eleven dimensional vielbein $\mathbb{V}^{\hat{a}}$ ($\hat{a} = 0, \ldots, 10$) is chosen of the form:

$$\mathbb{V}^r = e^{\alpha\phi}\, V^r\,; \qquad \mathbb{V}^a = V^a = \phi^a_I(\sigma^I - A^I)\,, \tag{2.1}$$

where σ^I is the basis of 1-forms on the twisted torus, which satisfy the Maurer-Cartan equations:

$$d\sigma^I + \frac{1}{2}\,\tau_{JK}{}^I\,\sigma^J \wedge \sigma^K = 0\,. \tag{2.2}$$

The vector fields A^I_μ are the Kaluza-Klein vectors. The internal metric G_{IJ} is given by:

$$G_{IJ} = \phi^a_I\,\phi_{Ja}\,. \tag{2.3}$$

The ansatz for the eleven dimensional 3-form is:

$$\hat{A}^{(3)} = \mathcal{A}^{(3)} + B_I \wedge V^I + A_{IJ} \wedge V^I \wedge V^J + C_{IJK}\, V^I \wedge V^J \wedge V^K\,, \tag{2.4}$$

where $\mathcal{A}^{(3)}$ is a four-dimensional (non-propagating) 3-form, $B_{I\mu\nu}$ are seven four-dimensional antisymmetric tensors, $A_{IJ\mu}$ are 21 vector fields and C_{IJK} are 35 scalar fields.

The ansatz for the eleven dimensional gravitino $\hat{\psi}$ reads:

$$\hat{\psi}(x, y) = (\Psi_A + \eta_{Aa}\, V^a) \otimes \mu^A\,, \tag{2.5}$$

where μ^A are spinors on the twisted torus. In these notations we will define $\eta_{AI} = \phi^a_I\,\eta_{Aa}$.

Supersymmetric Free Differential Algebra

Let us introduce the four dimensional curvatures, written as forms on superspace and expanded in the super-vielbein basis $\{V^r,\,\Psi_A\}$ (rheonomic parametrization [43]):

$$
\begin{aligned}
\mathscr{D}\phi^a_I &= \mathscr{D}_r\phi^a_I\,V^r - i\,\phi^b_I\,\overline{\eta}_{Ab}\gamma^5\Psi_B\,(\Gamma^a)_{AB}\,,\\
\hat{F}^I &\equiv dA^I + \frac{1}{2}\tau_{JK}{}^I\,A^J\wedge A^K + \frac{i}{2}\,\overline{\Psi}_A\gamma^5\,\wedge\Psi_B\,\Gamma^a_{AB}\,\phi^I_a\\
&\qquad \tilde{F}^I_{st}V^s\wedge V^t + i\,\phi^{Ia}\,\overline{\eta}_{Aa}\,\gamma^r\,\Psi_A\wedge V_r\,, \qquad (2.6)\\
F^{(4)} &\equiv d\mathcal{A}^{(3)} - B_I\wedge F^I - g_{IJKL}\,A^I\wedge A^J\wedge A^K\wedge A^L - \frac{1}{2}\,e^{2\alpha\phi}\,\overline{\Psi}_A\gamma^{rs}\wedge\Psi_A\wedge V_r\wedge V_s\\
&= \tilde{F}^{(4)}_{rstu}\,V^r\wedge V^s\wedge V^t\wedge V^u\,,\\
H_I &\equiv \mathscr{D}B_I + 2\,A_{JI}\wedge F^J + 4\,g_{IJKL}\,A^J\wedge A^K\wedge A^L + e^{\alpha\phi}\,\overline{\Psi}_A\gamma^5\,\gamma^r\,\wedge\Psi_B\,\Gamma^a_{AB}\,\phi_{Ia}\wedge V_r\\
&= \tilde{H}_{rstI}V^r\wedge V^s\wedge V^t + e^{2\alpha\phi}\,\overline{\Psi}_A\gamma^{rs}\,\eta_{AI}\wedge V_r\wedge V_s\,,\\
F^{(2)}_{IJ} &\equiv \mathscr{D}A_{IJ} - \frac{1}{2}\,\tau_{IJ}{}^K\,B_K - 3\,C_{IJK}\,F^K - 6\,g_{IJKL}\,A^K\wedge A^L - \frac{1}{2}\overline{\Psi}_A\wedge\Psi_B\,\Gamma^{ab}_{AB}\phi_{Ia}\phi_{Jb}\\
&= \tilde{F}^{(2)}_{stIJ}\,V^s\wedge V^t + 2\,e^{\alpha\phi}\overline{\Psi}_A\gamma^5\,\gamma^r\,\eta_{B[I}\,\phi_{J]a}\Gamma^a_{AB}\wedge V_r\,,\\
F^{(1)}_{IJK} &\equiv \mathscr{D}C_{IJK} - \tau_{[IJ}{}^L\,A_{K]L} + 4\,g_{IJKL}\,A^L = \tilde{F}^{(1)}_{rIJK}V^r + \overline{\Psi}_A\,\eta_{B[I}\,\phi_{Ja}\phi_{K]b}\Gamma^{ab}_{AB}\,,\\
F^{(0)}_{IJKL} &= -g_{IJKL} - \frac{3}{2}\,\tau_{[IJ}{}^M\,C_{KL]M} - \frac{1}{2}\,\overline{\eta}_{A[I}\,\eta_{BJ}\phi_{Ka}\phi_{L]b}\Gamma^{ab}_{AB}\,, \qquad (2.7)
\end{aligned}
$$

where antisymmetrization in the above formulas involve only the internal space indices $I, J, \ldots$. In (2.7) we have denoted by F^I the following quantities:

$$
F^I = dA^I + \frac{1}{2}\,\tau_{JK}{}^I\,A^J\wedge A^K\,. \qquad (2.8)
$$

The fermionic curvatures have the following parametrization:

$$
\begin{aligned}
\rho_A &= \mathcal{D}\Psi_A + \frac{1}{4}\,\phi^I_a\mathscr{D}\phi_{bI}\,(\Gamma^{ab})_{AB}\wedge\Psi_B\\
&= \tilde{\rho}_{rs}\,V^r\wedge V^s + \frac{\alpha}{2}\,\partial_r\phi\,\gamma^r{}_t\,V^t\wedge\Psi_A + \frac{i}{8}\,e^{\alpha\phi}\,\overline{\eta}_{Ca}\gamma_r\eta_{Cb}\,(\Gamma^{ab})_{AB}V^r\wedge\Psi_B +\\
&\quad + i\,e^{\alpha\phi}\,\phi^{Ia}\eta_{AI}\,\overline{\eta}_{Ba}\,\gamma_r\,\Psi_B\wedge V^r + \frac{1}{2}\,e^{-\alpha\phi}\,\tilde{F}^I_{st}\,\phi_{Ia}\,(\Gamma^a)_{AB}\,\gamma_5\gamma^s V^t\wedge\Psi_B -\\
&\quad - \frac{i}{2}\,(\overline{\eta}_{Ca}\gamma_r\Psi_C)\,\gamma^5\gamma^r(\Gamma^a)_{AB}\,\wedge\Psi_B - \frac{i}{4}\,\overline{\eta}_{Ca}\gamma^5\Psi_D(\Gamma_b)_{CD}\,(\Gamma^{ab})_{AB}\wedge\Psi_B -\\
&\quad - \frac{i}{2}\overline{\Psi}_B\gamma^5\Psi_C(\Gamma^a)_{BC}\,\eta_A - \frac{i}{24}\,e^{\alpha\phi}\,\tilde{F}^{(0)}_{abcd}(\Gamma^{abcd})_{AB}\gamma^r\Psi_B\wedge V^r +\\
&\quad + \frac{i}{3}\,\tilde{F}^{(1)}_{rabc}(\Gamma^{abc})_{AB}\left(\delta^r_s + \frac{1}{2}\,\gamma^r{}_s\right)\,\gamma^5\Psi_B\wedge V^s -\\
&\quad - i\,e^{-\alpha\phi}\,\tilde{F}^{(2)}_{rsab}(\Gamma^{ab})_{AB}\left(\gamma^r\,\delta^s_t + \frac{1}{4}\,\gamma^{rs}{}_t\right)\,\Psi_B\wedge V^t +\\
&\quad + i\,e^{-2\alpha\phi}\,\tilde{H}_{rsta}(\Gamma^a)_{AB}\left(\gamma^{rs}\delta^t_u + \frac{1}{6}\gamma^{rst}{}_u\right)\gamma^5\Psi_B\wedge V^u - \frac{i}{3}\,e^{-3\alpha\phi}\,\tilde{F}^{(4)}_{rstu}\gamma^{rst}\,\Psi_A\wedge V^u\,,\\
\rho_{Aa} &= \mathcal{D}\eta_{Aa} + \frac{1}{4}\,\phi^I_b\mathscr{D}\phi_{cI}(\Gamma^{bc})_{AB}\eta_{Ba}
\end{aligned}
$$

$$
\begin{aligned}
&= \tilde{\rho}_{Aa,r}\, V^r + \frac{1}{2}\, e^{-\alpha\phi}\, \phi^I_{(a}\mathscr{D}_r\phi_{b)I}(\Gamma^b)_{AB}\gamma^5\gamma^r\Psi_B + i\,\eta_{Ab}\,(\overline{\eta}_{Ba}\gamma^5\Psi_C)\,(\Gamma^b)_{BC} + \\
&\quad + \frac{i}{4}(\overline{\eta}_{Cb}\gamma_r\eta_{Ca})\gamma^5\gamma^r(\Gamma^b)_{AB}\Psi_A - \frac{i}{2}(\overline{\eta}_{Cb}\gamma_r\Psi_C)\gamma^5\gamma^r(\Gamma^b)_{AB}\eta_{Ba} + \\
&\quad + \frac{1}{4}\, e^{-2\alpha\phi}\phi_{Ia}\tilde{F}^I_{rs}\gamma^{rs}\,\Psi_A - \frac{i}{3}\tilde{F}^{(0)}_{bcde}\left[\Gamma^{bcd}\delta^e_a + \frac{1}{8}\Gamma^{bcde}{}_a\right]_{AB}\gamma^5\Psi_B - \\
&\quad - i\, e^{-\alpha\phi}\,\tilde{F}^{(1)}_{rbcd}\left[\Gamma^{bc}\,\delta^d_a + \frac{1}{6}\Gamma^{bcd}{}_a\right]_{AB}\gamma^r\Psi_B - \\
&\quad - i\, e^{-2\alpha\phi}\,\tilde{F}^{(2)}_{rsbc}\left[\Gamma^b\delta^c_a + \frac{1}{4}\Gamma^{bc}{}_a\right]_{AB}\gamma^5\gamma^{rs}\Psi_B - \frac{i}{3}\, e^{-3\alpha\phi}\,\tilde{H}_{rstb}\left[\delta^b_a + \frac{1}{2}\Gamma^b{}_a\right]_{AB}\gamma^{rst}\Psi_B - \\
&\quad - \frac{1}{24}\, e^{-4\alpha\phi}\,\tilde{F}^{(4)}_{rstu}\epsilon^{rstu}\Gamma_{a|AB}\Psi_B - \frac{1}{4}\omega_{bc,a}(\Gamma^{bc})_{AB}\Psi_B - \\
&\quad - \frac{i}{4}\overline{\eta}_{Cb}\gamma^5\Psi_D(\Gamma_c)_{CD}(\Gamma^{bc})_{AB}\eta_{Ba}\,. \qquad (2.9)
\end{aligned}
$$

In (2.6), (2.7) and (2.9) we have denoted by $\tilde{F}$, $\tilde{H}$, $\tilde{\rho}$ and $\tilde{\rho}_A$ the components in superspace of the curvatures along the space-time vielbeins. The *supercovariant* field strengths originate by projecting these components on the dx^μ basis.

The 7-form flux [44], which we shall denote by $\tilde{g}_{MNPQRST} = \tilde{g}\epsilon_{MNPQRST}$, transforms in the $\mathbf{1}_{+7}$ of $\mathrm{GL}(7,\mathbb{R})$ enters our discussion as an intregration constant which comes about when integrating the eleven dimensional field equation [35]:

$$
d(V_7\, P) = -\frac{1}{4}\, F^{(1)}_{IJK}\, F^{(0)}_{PQRS}\,\epsilon^{IJKPQRS}\,, \qquad (2.10)
$$

where

$$
P = \frac{1}{\sqrt{-g}}\,\epsilon^{\mu_1\ldots\mu_4}\,F^{(4)}_{\mu_1\ldots\mu_4}\,. \qquad (2.11)
$$

The fermion fields Ψ_A, η_{Aa} yield the four dimensional gravitino ψ_A and dilatino χ_{Aa} through the following combinations:

$$
\begin{aligned}
\psi_A &= e^{-\frac{1}{2}\,\alpha\phi}\,\Psi_A + \frac{1}{2}\, e^{\frac{1}{2}\,\alpha\phi}\,\gamma_r\gamma_5(\Gamma^a)_{AB}\eta_{Ba}\,V^r\,, \\
\chi_{Aa} &= e^{\frac{1}{2}\,\alpha\phi}\,\eta_{Aa}\,. \qquad (2.12)
\end{aligned}
$$

The above redefinitions are needed in order for the resulting kinetic term in the four dimensional lagrangian to be diagonal.

3. Local symmetries

Let us consider the eleven dimensional gauge transformation:

$$
\delta\hat{A}^{(3)} = d\hat{\Sigma}^{(2)}\,. \qquad (3.1)
$$

If we introduce the parameters $\Sigma^{(2)}$, $\Sigma^{(1)}_I$, $\Sigma^{(0)}_{IJ}$ through the following expansion:

$$
\hat{\Sigma}^{(2)} = \Sigma^{(2)} + \Sigma^{(1)}_I \wedge V^I + \Sigma^{(0)}_{IJ} \wedge V^I \wedge V^J\,, \qquad (3.2)
$$

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and the gauge parameter ω^I associated with the Kaluza-Klein vectors A^I, the lower dimensional theory is invariant under the following four dimensional tensor/vector-gauge transformations [15]:

$$\begin{aligned}
\delta \mathcal{A}^{(3)} &= d\Sigma^{(2)} + \Sigma^{(1)}_I \wedge F^I \,, \\
\delta B_I &= \mathscr{D}\Sigma^{(1)}_I + 2\,\Sigma^{(0)}_{IJ}\, F^J + \omega^K\, \tau_{KI}{}^N\, B_N - 12\, g_{IJKL}\omega^J\, A^K \wedge A^L \,, \\
\delta A_{IJ} &= \frac{1}{2}\,\tau_{IJ}{}^K\,\Sigma^{(1)}_K + \mathscr{D}\Sigma^{(0)}_{IJ} - 2\,\omega^N\,\tau_{N[I}{}^K\, A_{J]K} + 12\, g_{IJKL}\omega^K\, A^L \,, \\
\delta A^I &= \mathscr{D}\omega^I \,, \\
\delta C_{IJK} &= -\Sigma^{(0)}_{M[I}\,\tau_{JK]}{}^M + 3\,\omega^L\,\tau_{L[I}{}^M\, C_{JK]M} - 4\, g_{IJKL}\,\omega^L \,.
\end{aligned} \tag{3.3}$$

We have denoted by $\mathscr{D}$ the covariant derivative with respect to the gauge group with parameters ω^I:

$$\mathscr{D}T_{I_1\ldots I_k} \equiv dT_{I_1\ldots I_k} + (-1)^k\, k\, A^L\,\tau_{L[I_1}{}^K\, T_{I_2\ldots I_k]K} \,. \tag{3.4}$$

We shall use a different symbol $\mathcal{D}$ to denote the covariant derivative with respect to the spin connection.

Under the gauge transformations (3.3) the field strengths in (2.6) and (2.7) transform covariantly. In particular they are invariant under the transformations parametrized by the Σ-parameters, while transform covariantly under those parametrized by ω^I. The supersymmetry variation of the various fields are computed by contracting the correponding curvatures by the supersymmetry parameter ϵ^A and keeping in mind that:

$$i_{\epsilon^A}\,\psi_B \equiv \psi_B(\epsilon^A) = \delta^A_B \,, \tag{3.5}$$

except for the gravitino for which the supersymmetry transformation is computed as [43]:

$$\delta\psi_A = \mathcal{D}\epsilon_A - \frac{1}{4}\,\phi^I_a \mathscr{D}_r \phi_{Ib}(\Gamma^{ab})_{AB}\,\epsilon_B\, V^r + i_\epsilon \rho_A \,. \tag{3.6}$$

As an example let us compute the variation of the Kaluza-Klein vectors keeping just the two-fermion terms, using the parametrization of the corresponding field strength given in (2.6) and taking into account (2.12):

$$\delta A^I_\mu = i_\epsilon F^I = i\, e^{\alpha\phi}\,\phi^{Ia}\overline{\chi}_{Aa}\,\gamma_r \epsilon_A\, V^r_\mu - i\, e^{\alpha\phi}\overline{\epsilon}_A\gamma^5\left[\psi_{B\mu} - \frac{1}{2}\gamma_s\gamma_5(\Gamma^b)_{BC}\,\chi_{Cb}\, V^s_\mu\right](\Gamma^a)_{AB}\phi^I_a \,. \tag{3.7}$$

Let us now consider the local supersymmetry transformations as deduced from the rheonomic parametrization of the four dimensional curvatures.

$$\delta C_{IJK} = \overline{\epsilon}^A(\Gamma^{ab})_{AB}\chi_{B[I}\phi_{aJ}\,\phi_{bK]} \,, \tag{3.8}$$

$$\delta\phi^a_I = -i\overline{\chi}_{AI}\gamma^5\epsilon_B(\Gamma^a)_{AB} \,, \tag{3.9}$$

$$\delta B_I = \overline{\epsilon}_A\gamma_{rs}\chi_{AI}V^r \wedge V^s - 2e^{2\alpha\phi}\overline{\epsilon}_A\gamma^5\gamma_r\left[\psi_B - \frac{1}{2}\gamma_s\gamma_5(\Gamma^b)_{BC}\chi_{Cb}V^s\right](\Gamma^a)_{AB}\phi_{Ia} \wedge V^r +$$

$$\begin{aligned}
&+2i\, e^{\alpha\phi} A_{IJ} \wedge g^{JK} \overline{\chi}_{AK}\, \gamma_r \epsilon_A\, V^r - \\
&-2i\, e^{\alpha\phi} A_{IJ} \wedge \overline{\epsilon}_A \gamma^5 \left[\psi_B - \frac{1}{2}\gamma_s\gamma_5(\Gamma^b)_{BC}\chi_{Cb}\, V^s\right] (\Gamma^a)_{AB}\phi^J_a\,, \qquad (3.10)\\
\delta A_{IJ} &= 2\, e^{\alpha\phi}\overline{\epsilon}_A\gamma^5\gamma_r\,\chi_{B[I}\phi_{J]a}(\Gamma^a)_{AB}\, V^r + e^{\alpha\phi}\,\overline{\epsilon}_A\psi_B\,(\Gamma^{ab})_{AB}\,\phi_{Ia}\phi_{Jb} + \\
&+3i\, e^{\alpha\phi}\, C_{IJK}\, g^{KM}\overline{\chi}_{AM}\,\gamma_r\epsilon_A\, V^r - \\
&-3i\, e^{\alpha\phi} C_{IJK}\overline{\epsilon}_A\gamma^5 \left[\psi_B - \frac{1}{2}\gamma_s\gamma_5(\Gamma^b)_{BC}\chi_{Cb}\, V^s\right] (\Gamma^a)_{AB}\phi^K_a\,, \qquad (3.11)\\
\delta A^I &= i\, e^{\alpha\phi}\, g^{IM}\overline{\chi}_{AM}\,\gamma_r\epsilon_A\, V^r - i\, e^{\alpha\phi}\overline{\epsilon}_A\gamma^5 \left[\psi_B - \frac{1}{2}\gamma_s\gamma_5(\Gamma^b)_{BC}\chi_{Cb}\, V^s\right] (\Gamma^a)_{AB}\phi^I_a\,, \qquad (3.12)\\
\delta\mathcal{A}^{(3)} &= e^{3\alpha\phi}\overline{\epsilon}_A\gamma_{rs}\left[\psi_A - \frac{1}{2}\gamma_s\gamma_5(\Gamma^b)_{AB}\chi_{Bb}\, V^s\right] \wedge V^r \wedge V^s + \\
&+i\, e^{\alpha\phi}\, B_I \wedge g^{IM}\overline{\chi}_{AM}\,\gamma_r\epsilon_A\, V^r - \\
&-i\, e^{\alpha\phi}\, B_I \wedge \overline{\epsilon}_A\gamma^5 \left[\psi_B - \frac{1}{2}\gamma_s\gamma_5(\Gamma^b)_{BC}\chi_{Cb}\, V^s\right] (\Gamma^a)_{AB}\phi^I_a\,. \qquad (3.13)
\end{aligned}$$

In the above formulas ${\phi_I}^a$ are the metric moduli of the internal manifold and span the coset manifold $\mathrm{GL}(7,\mathbb{R})/\,\mathrm{SO}(7)$. the scalar ϕ is related to the volume V_7 of the torus:

$$V_7 = \det(V_7) = e^{-2\alpha\phi}\,. \qquad (3.14)$$

The value of α is tipically fixed to $7/3$ in order for the bosonic fields to have the standard grading with respect to the $\mathrm{O}(1,1)$ rescaling of V_7:

$$\phi \to \phi - \beta\,. \qquad (3.15)$$

The grading of the various fields, for $\alpha = 7/3$, can be computed to be:

$$\begin{aligned}
&A^I\left[-\frac{9}{7}\alpha = -3\right]; \quad A_{IJ}\left[-\frac{3}{7}\alpha = -1\right]; \quad B_I\left[-\frac{12}{7}\alpha = -4\right]; \quad \phi^a_I\left[\frac{2}{7}\alpha = \frac{2}{3}\right];\\
&V^I\left[-\frac{9}{7}\alpha = -3\right]; \quad V^a\left[-\alpha = -\frac{3}{7}\right]; \quad {\tau_{IJ}}^K\left[\frac{9}{7}\alpha = +3\right]; \quad g_{IJKL}\left[\frac{15}{7}\alpha = +5\right];\\
&\Psi_A\left[-\frac{1}{2}\alpha = -\frac{7}{6}\right]; \quad \eta_{Aa}\left[\frac{1}{2}\alpha = \frac{7}{6}\right]; \quad \psi_A[0]; \quad \chi_{Aa}[0];\\
&e^{k\alpha\phi}\left[-k\,\alpha = -\frac{7}{3}\,k\right]. \qquad (3.16)
\end{aligned}$$

As far as the spinor fields are concerned, their supersymmetry transformation rules, up to three fermion terms, are:

$$\begin{aligned}
\delta\psi_A &= \mathscr{D}\epsilon_A + \frac{1}{4}\,\phi^I_a\mathscr{D}_r\phi_{bI}\,(\Gamma^{ab})_{AB}\epsilon_B\, V^r - \frac{1}{8}\, e^{-\alpha\phi}\, F^I_{st}\,\phi_{Ia}\,(\Gamma^a)_{AB}\gamma_5\gamma^{st}\gamma_r V^r\epsilon_B - \\
&+\frac{i}{16}\, e^{\alpha\phi}F^{(0)}_{abcd}(\Gamma^{abcd})_{AB}\gamma_r\, V^r\epsilon_B - \frac{i}{2}\, F^{(1)}_{abcr}(\Gamma^{abc})_{AB}\gamma^5\, V^r\epsilon_B + \\
&-\frac{3i}{8}\, e^{-\alpha\phi}\, F^{(2)}_{abrs}\,(\Gamma^{ab})_{AB}{\gamma^{rs}}_t\, V^t\epsilon_B + \frac{i}{4}\, e^{-\alpha\phi}\, F^{(2)}_{abrs}\,(\Gamma^{ab})_{AB}\gamma^r\, V^s\epsilon_B - \\
&-\frac{i}{2}\, e^{-2\alpha\phi}\, H_{arst}(\Gamma^a)_{AB}\gamma^5{\gamma^{rst}}_u\, V^u\,\epsilon_B + \frac{i}{4}\, e^{-3\alpha\phi}F^{(4)}_{rstu}\gamma^{rst}\, V^u\epsilon_A -
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{8}\,e^{\alpha\phi}\,\omega_{ab,c}(\Gamma^c\Gamma^{ab})_{AB}\gamma_5\gamma_r\,V^r\,\epsilon_B\,.\\
\delta\chi_{Aa} \;=\;& \frac{1}{2}\phi^I_{(a}\mathscr{D}_r\phi_{b)I}(\Gamma^b)_{AB}\gamma^5\gamma^r\epsilon_B+\frac{1}{4}\,e^{-\alpha\phi}F^I_{st}\,\phi_{Ia}\gamma^{st}\epsilon_A-\\
&-\frac{i}{3}\,e^{\alpha\phi}\,F^{(0)}_{bcde}\left[\Gamma^{bcd}\,\delta^e_a+\frac{1}{8}\Gamma^{bcde}{}_a\right]_{AB}\gamma^5\epsilon_B+i\,F^{(1)}_{bcdr}\left[\Gamma^{bc}\delta^d_a+\frac{1}{6}\Gamma^{bcd}{}_a\right]_{AB}\gamma^r\epsilon_B-\\
&-i\,e^{-\alpha\phi}F^{(2)}_{bcrs}\left[\Gamma^b\delta^c_a+\frac{1}{4}\Gamma^{bc}{}_a\right]_{AB}\gamma^5\gamma^{rs}\epsilon_B+\frac{i}{3}\,e^{-2\alpha\phi}H_{brst}\left[\delta^b_a+\frac{1}{2}\Gamma^b{}_a\right]_{AB}\gamma^{rst}\epsilon_B-\\
&-\frac{1}{24}\,e^{-3\alpha\phi}F^{(4)}_{rstu}\epsilon^{rstu}\Gamma_{a|AB}\epsilon_B-\\
&-\frac{1}{4}\,e^{\alpha\phi}\,\omega_{bc,a}\,(\Gamma^{bc})_{AB}\,\epsilon_B\,,
\end{aligned}
\tag{3.17}
$$

where we have defined:

$$
\begin{aligned}
\omega_{ab,c} &= d_{a,bc}+d_{b,ca}-d_{c,ab}\\
d_{a,bc} &= \frac{1}{2}\,\tau_{bc,a}+\frac{i}{2}\,e^{-\alpha\phi}\,\overline{\chi}_{Ab}\,\gamma^5\chi_{Bc}\,\Gamma_{a|AB}\quad;\quad \tau_{bc,a}=\phi^I_b\phi^J_c\phi_{Ka}\,\tau_{IJ}{}^K\,.
\end{aligned}
\tag{3.18}
$$

In the transformation rules (3.17) we used the components of the ordinary field strengths F, H in place of the components of the corresponding supercovariant field strengths $\tilde{F}$, $\tilde{H}$ since the final expressions would differ by three fermion terms.

4. The embedding tensor description applied to different classes of gaugings

In standard four dimensional maximal supergravity the electric e_Λ and magnetic m^Λ charges, where $\Lambda = 1,\ldots,28$, transform together in the **56** of $E_{7(7)}$ and the most general gauging can be described in terms of an embedding tensor [37–41] $\theta_n{}^\sigma \equiv \{\theta_\Lambda{}^\sigma,\,\theta^{\Lambda\sigma}\}$ ($n = 1,\ldots,56$ and $\sigma = 1,\ldots,133$), which expresses the generators X_n of the gauge algebra $\mathfrak{g}$ in terms of $E_{7(7)}$ generators t_σ:

$$
X_n \,=\, \theta_n{}^\sigma\,t_\sigma\,. \tag{4.1}
$$

In this notation consistency of the gauging requires the rank of θ, as a 56×133 matrix, not to be greater than 28 since no more that 28 gauge vectors can take part to the minimal couplings. Supersymmetry and closure of the gauge algebra inside $E_{7(7)}$ require a linear and a quadratic condition in θ respectively [37, 38]:

$$
\theta \in \mathbf{912} \subset \mathbf{56}\times\mathbf{133}\,, \tag{4.2}
$$

$$
\theta^{\Lambda[\sigma}\,\theta_\Lambda{}^{\gamma]} = 0\,. \tag{4.3}
$$

The above constraints are all $E_{7(7)}$ covariant. The last condition ensures that *there always exists a symplectic rotation acting on the index Λ (electric and magnetic) as a consequence of which all the vectors associated with the generators X_n are electric (or all magnetic).*

Once θ is fixed, as a solution of (4.2) and (4.3), the structure of the gauge algebra is also fixed. Indeed if we introduce the following $E_{7(7)}$-tensor:

$$
X_{mn}{}^p \,=\, \theta_m{}^\sigma\,(t_\sigma)_n{}^p\,, \tag{4.4}
$$

the gauge algebra has the following structure:

$$[X_m, X_n] = -X_{mn}{}^p X_p\,. \tag{4.5}$$

In terms of $X_{mn}{}^p$, the **912** in the decomposition of $\mathbf{56}\times\mathbf{133}$ is singled out by requiring the constraint $X_{(mnp)} = 0$ (the indices in the **56** are raised and lowered by using the symplectic invariant matrix), which can be taken as an equivalent formulation of condition (4.2).

In the standard formulation of gauged supergravity, the definition of the gauged lagrangian is always referred to the symplectic frame (the *electric frame*) in which the components of θ are all electric (namely in which $\theta^{\Lambda\sigma} = 0$) so that only the electric vector fields A^Λ_μ are involved in the gauging. Given a generic solution $\theta^{\Lambda\sigma}$, $\theta_\Lambda{}^\sigma$ of eqs. (4.2) and (4.3), the electric frame is reached by means of a symplectic rotation whose existence, as previously stressed, is guaranteed by eq. (4.3), and which maps:

$$\theta_n{}^\sigma \equiv \{\theta_\Lambda{}^\sigma,\, \theta^{\Lambda\sigma}\} \to \theta'_n{}^\sigma \equiv \{\theta'_\Lambda{}^\sigma,\, 0\}\,. \tag{4.6}$$

In this frame the gauge connection will have the form $\Omega_\mu = A^\Lambda_\mu\,\theta'_\Lambda{}^\sigma\, t_\sigma$. In a recent work [41] a novel formulation of gauged supergravity was proposed in which the lagrangian can be written in a generic symplectic frame, as function of both electric ($\theta_\Lambda{}^\sigma$) and magnetic ($\theta^{\Lambda\sigma}$) charges, coupled in a symplectic-invariant way to (non-abelian) electric (A^Λ_μ) and magnetic $A_{\mu\Lambda}$ gauge fields. The new gauge connection now reads:

$$\Omega_\mu = A^\Lambda_\mu\, X_\Lambda + A_{\mu\Lambda}\, X^\Lambda\,. \tag{4.7}$$

This formulation requires the introduction of 133 tensor fields $B_{\mu\nu\sigma}$ in the adjoint of $\mathrm{E}_{7(7)}$ which enter the lagrangian only in the combination with the magnetic charges: $\theta^{\Lambda\sigma}\, B_{\mu\nu\sigma}$. The advantage of this formulation is that the $\mathrm{E}_{7(7)}$-covariance of the field equations and Bianchi identities is still manifest (provided θ is transformad together with all the other fields). The new fields $B_{\mu\nu\sigma}$ and $A_{\mu\Lambda}$ are described in such a way as not to introduce any new propagating degree of freedom. This is reflected by the fact that the corresponding field equations (namely the equations obtained by varying the lagrangian with respect to the two type of fields) are non-dynamical. In addition to this we have vector and tensor-gauge invariance of the lagrangian, which allow us, by performing various kind of gauge fixing, to distribute the 128 propagating bosonic degrees of freedom among all the bosonic fields of the theory. We refer the reader to [41] for the explicit form of the field equations and the gauge transformation laws.

A particular case in which the non-dynamical field equations are easily solvable is the case in which the magnetic components of the embedding tensor contract only isometries t_i of the scalar manifold which act as translations on a set of corresponding scalar fields φ^i:

$$t_i: \qquad \varphi^i \to \varphi^i + c^i\,. \tag{4.8}$$

The index Λ naturally splits in the couple of indices $\Lambda = \mathcal{I}, U$, so that $\theta^{\mathcal{I}i}$ is a non-singular square matrix. In this basis we have:

$$\begin{aligned}
\theta^{\mathcal{I}\sigma} &= 0\,,\ \sigma\neq i\,; \qquad \theta^{U\,\sigma} = 0\,,\ \forall\,\sigma\,,\\
\theta^{\mathcal{I}[i}\,\theta_{\mathcal{I}}{}^{j]} &= 0\,,
\end{aligned} \tag{4.9}$$

and therefore only the tensors $B_{\mu\nu i}$ enter the lagrangian. Let us consider two ways of gauge fixing the vector/tensor gauge invariance which are relevant to our analysis. We can use the gauge invariance of $A_{\mathcal{I}\mu}$ to eliminate the scalar fields φ^i, recalling that the covariant derivative on these scalar fields read:

$$D_\mu\varphi^i = \partial_\mu\varphi^i + \theta^{\mathcal{I}i}\, A_{\mu\mathcal{I}} + \dots\,, \tag{4.10}$$

where the ellipses refer to the electric minimal coupings which are not relevant to our discussion. Then we use one of the non-dynamical equations of motion to eliminate $A_{\mu\mathcal{I}}$ in favor of $B_{\mu\nu i}$. The resulting gauge-fixed action will describe the tensors $B_{\mu\nu i}$ instead of the scalars φ^i and the electric vectors $A^\Lambda_\mu = \{A^{\mathcal{I}}_\mu,\, A^U_\mu\}$. In these models the original second order constraint (4.3) becomes eq. (4.9) which has the form of the $e \times m = 0$ constraint found in the literature when dealing with supergravity theories coupled to antisymmetric tensor fields [45 – 49].

On the other hand we could start by fixing the tensor-gauge invariance by eliminating the electric fields $A^{\mathcal{I}}_\mu$ through their coupling terms with the tensors $B_{\mu\nu i}$:

$$F^{\mathcal{I}}_{\mu\nu} + \frac{1}{2}\,\theta^{\mathcal{I}i}\, B_{\mu\nu i}\,. \tag{4.11}$$

Then we can use one of the non-dynamical field equations to eliminate $B_{\mu\nu i}$ in favor of the remaining gauge fields. The resulting theory will describe 70 scalar fields, no antisymmetric tensor field and the new electric vectors $A_{\mu\mathcal{I}}$, A^U_μ. This procedure has thus automatically produced the symplectic transformation which brought our original θ to the electric frame.

First example: M-theory compactification on twisted tori with flux

If we are considering toroidal compactifications of eleven dimensional supergravity to four dimensions, the higher dimensional origin of the four dimensional fields is specified by branching the relevant $\mathrm{E}_{7(7)}$-representations with respect to the $\mathrm{GL}(7,\mathbb{R})$ subgroup associated with the metric moduli of the seven-torus:

$$\mathbf{56} \rightarrow \overline{\mathbf{7}}_{-3} + \mathbf{21}_{-1} + \overline{\mathbf{21}}_{+1} + \mathbf{7}_{+3}\,, \tag{4.12}$$

$$\mathbf{133} \rightarrow \mathbf{7}_{-4} + \overline{\mathbf{7}}_{+4} + \overline{\mathbf{35}}_{-2} + \mathbf{35}_{+2} + \mathbf{48}_0 + \mathbf{1}_0\,, \tag{4.13}$$

$$\begin{aligned}\mathbf{912} \rightarrow\ & \mathbf{1}_{-7} + \mathbf{1}_{+7} + \mathbf{35}_{-5} + \overline{\mathbf{35}}_{+5} + (\overline{\mathbf{140}} + \overline{\mathbf{7}})_{-3} + (\mathbf{140} + \mathbf{7})_{+3} + \mathbf{21}_{-1} + \overline{\mathbf{21}}_{+1} + \\ & + \mathbf{28}_{-1} + \overline{\mathbf{28}}_{+1} + \mathbf{224}_{-1} + \overline{\mathbf{224}}_{+1}\,.\end{aligned} \tag{4.14}$$

In the branching of the $\mathbf{56}$ the $\overline{\mathbf{7}}_{-3}$ and $\mathbf{21}_{-1}$ define A^I_μ, $A_{\mu IJ}$ respectively while $\mathbf{7}_{+3}$ and $\overline{\mathbf{21}}_{+1}$ their magnetic duals. In the branching of the adjoint representation of $\mathrm{E}_{7(7)}$ we denote by ${t_M}^N$, t^{MNP}, t_P the generators in the $\mathbf{48}_0 + \mathbf{1}_0$, $\mathbf{35}_{+2}$ and $\overline{\mathbf{7}}_{+4}$ respectively (with an abuse of notation we characterize each generator by the representation of the corresponding parameter, this allows a simpler interpretation of the table below). In the solvable Lie algebra representation of the scalar manifold, the metric moduli ϕ^a_I parametrize the generators ${t_I}^J$ with $I \geq J$, while the scalars C_{IJK} and $\tilde{\phi}^I$ (dual to $B_{\mu\nu I}$) are parameters of the generators t^{MNP} and t_M repsectively. The $\mathbf{912}$ is the representation of the embeddig matrix, which encodes all possible deformations (minimal couplings, mass terms) of the ungauged

$N = 8$ theory. It is natural therefore to identify the background quantities $\tau_{IJ}{}^K$, g_{IJKL}, $\tilde{g}$ with components of the embedding tensor. Indeed each component representation on the right hand side of (4.14) defines a consistent gauged supergravity. Some of them have an immediate interpretation in terms of background fluxes, like $\overline{\mathbf{35}}_{+5}$ which represents the 4-form flux g_{IJKL}, or parameters related to the geometry of the internal space, like the $\mathbf{140}_{+3}$, which defines the twist-tensor $\tau_{MN}{}^P$. The structure of the gauge algebra implied by each of these representations is best understood from table:

	$\mathbf{7}_{+3}$	$\overline{\mathbf{21}}_{+1}$	$\mathbf{21}_{-1}$	$\overline{\mathbf{7}}_{-3}$
$\overline{\mathbf{7}}_{+4}$	$\mathbf{1}$	$\overline{\mathbf{35}}$	$\mathbf{140}+\mathbf{7}$	$\overline{\mathbf{28}}+\overline{\mathbf{21}}$
$\mathbf{35}_{+2}$	$\overline{\mathbf{35}}$	$\mathbf{140}$	$\overline{\mathbf{21}}+\overline{\mathbf{224}}$	$\mathbf{21}+\mathbf{224}$
$\mathbf{48}_0$	$\mathbf{140}+\mathbf{7}$	$\overline{\mathbf{21}}+\overline{\mathbf{28}}+\overline{\mathbf{224}}$	$\mathbf{21}+\mathbf{28}+\mathbf{224}$	$\overline{\mathbf{140}}+\overline{\mathbf{7}}$
$\mathbf{1}_0$	$\mathbf{7}$	$\overline{\mathbf{21}}$	$\mathbf{21}$	$\overline{\mathbf{7}}$
$\overline{\mathbf{35}}_{-2}$	$\overline{\mathbf{21}}+\overline{\mathbf{224}}$	$\mathbf{21}+\mathbf{224}$	$\overline{\mathbf{140}}$	$\mathbf{35}$
$\mathbf{7}_{-4}$	$\mathbf{28}+\mathbf{21}$	$\overline{\mathbf{140}}+\overline{\mathbf{7}}$	$\mathbf{35}$	$\mathbf{1}$

The first row and column contain the representations in the branchings of $\mathbf{56}$ and the $\mathbf{133}$ respectively, while the bulk contains representations in the branching of $\mathbf{912}$. The table specifies the origin of the latter representations in the branching of the product $\mathbf{56}\times\mathbf{133}$ and it should be read as "first row times first column gives bulk". The grading of each entry of the table has been suppressed for the sake of simplicity, since it coincides with the sum of the gradings of the corresponding elements in the first row and column.

The gauged supergravity models describing the class of compactifications we are considering are thus obtained by restricting θ to the representations $\mathbf{140}_{+3}+\overline{\mathbf{35}}_{+5}+\mathbf{1}_{+7}$. If we define on the $\mathbf{56}$ representation the following symplectic product:

$$V^n\,W^m\,\mathbb{C}_{nm} = V^M\,W_M - V_M\,W^M + V_{NM}\,W^{NM} - V^{NM}\,W_{NM}\,, \tag{4.15}$$

the relevant components of the tensor $X_{mn}{}^p$ read:

$$\begin{aligned}
X_{MN}{}^{PQ}{}_R &= X_{MN,R}{}^{PQ} = -\tau_{MN}{}^{[P}\delta_R^{Q]}\\
X^{MN,PQ}{}_R &= X^{MN}{}_R{}^{PQ} = -\frac{1}{2}\,\delta_R^{[P}\,\epsilon^{Q]MNN_1N_2N_3N_4}\,g_{N_1N_2N_3N_4}\,,\\
X^{MN}{}_{PQ,R} &= -X^{MN}{}_{R,PQ} = 3\,\tau_{[PQ}{}^{[M}\delta_{R]}^{N]}\,,\\
X^{MN,RS,LT} &= X^{MN,LT,RS} - 3\,\tau_{PQ}{}^{[M}\,\epsilon^{N]PQRSLT}\,,\\
X_M{}^L{}_S &= -X_{M,S}{}^L = \tau_{MS}{}^L\,,\\
X_M{}^{RS}{}_{PQ} &= -X_{M,PQ}{}^{RS} = 2\,\tau_{M[P}{}^{[R}\delta_{Q]}^{S]}\,,\\
X_{M,PQ,R} &= -X_{M,R,PQ} = g_{MPQR}\,,\\
X_M{}^{PQ,RS} &= X_M{}^{RS,PQ} = -g_{MM_1M_2M_3}\,\epsilon^{M_1M_2M_3PQRS}\,,\\
X_M{}^{PQ}{}_N &= X_{M,N}{}^{PQ} = -\tilde{g}\,\delta_{MN}^{PQ}\,.
\end{aligned} \tag{4.16}$$

One can verify that $X_{(mnp)} = 0$, consistently with the $\mathbf{912}$ condition. If we apply to this model the construction outlined above, we can write a consistent theory (with manifest $\mathrm{E}_{7(7)}$ global on-shell covariance) which describes among the other fields the tensors $B_{\mu\nu M}$

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and their dual scalar fields $\tilde{\phi}^M$ at the same time. Since in this case the magnetic components of θ are just $\theta_{MN}{}^P = -(3/2)\,\tau_{MN}{}^P$, if we denote by r the rank of this 21×7 matrix, the only tensor fields entering the lagrangian will be r out of the tensors $B_{\mu\nu P}$. The gauge connection has the form:

$$\Omega_\mu = \tilde{A}_\mu^{MN}\, X_{MN} + A_{\mu MN}\, X^{MN} + A_\mu^M\, X_M\,, \tag{4.17}$$

and involves also the magnetic vector fields $\tilde{A}_\mu^{MN}$. To obtain the model describing the tensor fields $B_{\mu\nu I}$ in place of their dual scalar fields, namely the model discussed in the first sections of the present paper, we can fix the gauge invariance associated with $\tilde{A}_\mu^{MN}$ to eliminate $\tilde{\phi}^M$, through the magnetic minimal coupling:

$$D_\mu\tilde{\phi}^M = \partial_\mu\tilde{\phi}^M + \tau_{PQ}{}^M\,\tilde{A}_\mu^{PQ} + \dots\,, \tag{4.18}$$

where the ellipses denote the electric minimal couplings. Then we can use one of the non-dynamical field equations, which reads:

$$\tau_{PQ}{}^M\,\epsilon^{\mu\nu\rho\sigma}\,(\partial_\nu B_{\rho\sigma M} + \dots) \propto G_{RS}\,\tau_{PQ}{}^R\,(\tau_{NL}{}^S\,\tilde{A}^{NL\mu} + \dots)\,, \tag{4.19}$$

to eliminate $\tilde{A}_\mu^{MN}$ in favor of $B_{\rho\sigma M}$ in the lagrangian. The ellipses on the left hand side of (4.19) denote topological terms involving the electric vector fields, while the ellipses on the right hand side stand for electric minimal couplings.

We could have proceeded differently by first fixing the tensor gauge invariance associated with $B_{\mu\nu M}$ in order to eliminate r of the $A_{\mu MN}$ fields and then expressing the tensor fields in terms of the remaining vector fields through one of the non-dynamical field equations. This procedure would have yield the gauged $N = 8$ supergravity with no tensor field and 70 scalar fields considered in [30]. This clarifies the relation between the dual descriptions of M-theory compactification on twisted tori with fluxes studied in the literature, namely the model with tensor fields in which the local symmetries are encoded in a (supersymmetric) free differential algebra, and the dual model without tensor fields in which the local symmetries of the lagrangian is described by an ordinary Lie algebra.

Second example: the $\mathrm{CSO}(p,q,r)$ **gauging as an M-theory compactification.**

From the branching of the **912** with respect to $\mathrm{GL}(7,\mathbb{R})$, we may consider the gauged model arising from the components:

$$\theta_{MN} = \theta_{(MN)} \in \mathbf{28}_{-1}\,; \qquad \tau_M \in \mathbf{7}_{+3}\,; \qquad \tilde{g} \in \mathbf{1}_{+7}\,, \tag{4.20}$$

$\tilde{g}$ being the 7-form flux. These entries of the embedding tensor can be re-arranged in a single 8×8 symmetric tensor θ_{AB} $(A, B = 1,\dots,8 = 1, M)$ transforming in the **36** of $\mathrm{SL}(8,\mathbb{R})$, maximal subgroup of $\mathrm{E}_{7(7)}$:

$$\theta_{AB} = \begin{pmatrix} \tilde{g} & \tau_N \\ \tau_M & \theta_{MN} \end{pmatrix}. \tag{4.21}$$

Indeed with respect to $\mathrm{GL}(7,\mathbb{R})$ the following branching holds:

$$\mathbf{36} \to \mathbf{28}_{-1} + \mathbf{7}_{+3} + \mathbf{1}_{+7}\,. \tag{4.22}$$

The corresponding gauge algebra generators have the form $X_{AB} = \{X_I,\, X_{IJ}\}$ and are gauged by the vectors $A_\mu^{AB} = \{A_\mu^I,\, \tilde{A}_\mu^{IJ}\}$ in the $\mathbf{28}'$ of $\mathrm{SL}(8,\mathbb{R})$. They close the following algebra:

$$[X_{AB}, X_{CD}] = f_{AB,CD}{}^{EF}\, X_{EF} \tag{4.23}$$
$$f_{AB,CD}{}^{EF} = 2\,\delta_{[A}^{[E}\theta_{B][C}\delta_{D]}^{F]}\,. \tag{4.24}$$

These are the well known $\mathrm{CSO}(p,q,r)$ gaugings originally constructed in [50], where p,q,r $(p+q+r=8)$ represent the number of positive, negative and null eigenvalues of θ_{AB} . For $p=8, q=0, r=0$ we have the SO(8) gauging constructed by de Wit and Nicolai [29]. This gauging is realized by setting $\theta_{AB} = \delta_{AB}$ namely $\theta_{IJ} = \delta_{IJ}$, $\tau_I = 0$, $\tilde{g} = 1$. In general the group $\mathrm{CSO}(p,q,r)$ describes the isometries of the following seven dimensional hypersurface embedded in $\mathbb{R}^8$ [51–53]:

$$\theta_{AB}\, z^A\, z^B = R^2\,. \tag{4.25}$$

and which can be written locally as the product $\mathscr{H}^{p,q}\times\mathbb{R}^r$, $\mathscr{H}^{p,q}$ being a $(p+q)$-dimensional hyperboloid. The Maurer-Cartan equations for this manifold have the form:

$$dV^I + \omega^{IJ}\,\theta_{JK}\wedge V^K - \tau_J\, V^J\wedge V^I = 0\,, \tag{4.26}$$
$$d\omega^{IJ} + \omega^{IK}\theta_{KL}\wedge\omega^{LJ} + 2\,\omega^{K[I}\,\tau_K\,\wedge V^{J]} - \tilde{g}\, V^I\wedge V^J = 0\,, \tag{4.27}$$

V^I and ω^{IJ} being the vielbein and the connection of the manifold.

Together with the components (4.20) we may switch on also the flux g_{IJKL} in the $\overline{\mathbf{35}}_{+5}$. The connection of the gauge algebra now becomes:

$$\Omega_\mu = A_\mu^I\, X_I + \tilde{A}_\mu^{IJ}\, X_{IJ} + A_{IJ\mu}\, X^{IJ}\,. \tag{4.28}$$

Let us consider the following basis of $\mathrm{E}_{7(7)}$ generators:

$$\{t_M{}^N,\, t^{MNP},\, t_{MNP},\, t^P,\, t_P\}\,. \tag{4.29}$$

We refer the reader to the appendix for the complete commutation relations among the above generators. With respect to the basis (4.29) the gauge generators have the following expression:

$$\begin{aligned} X_{MN} &= -\theta_{[M|P}\, t_{N]}{}^P - \tau_{[M}\, t_{N]}\,, \\ X_M &= \frac{1}{2}\,(\tilde{g}\, t_M + \tau_N\, t_M{}^N + \theta_{MN}\, t^N) + 2\, g_{MNPQ}\, t^{NPQ} + \frac{1}{14}\,\tau_M\, t\,, \\ X^{MN} &= 3\, g_{PQRS}\,\epsilon^{PQRSMNT}\, t_T\,. \end{aligned} \tag{4.30}$$

where we have denoted by t the $\mathrm{O}(1,1)$ generator $t_M{}^M$. For this gauging the second order condition (4.3) read:

$$\theta_{M[N}\, g_{PQRS]} = 0\,; \qquad \tau_{[N}\, g_{PQRS]} = 0\,, \tag{4.31}$$

which imply that the X^{MN} generators satisfy the following constraints:

$$\theta_{MN} X^{NP} = 0 \,; \qquad \tau_N \, X^{NP} = 0 \,. \tag{4.32}$$

Conditions (4.31) guarantee that the generators in (4.30) close an algebra, which can be found to have the following structure:

$$\begin{aligned} [X_{MN},\, X_{PQ}] &= \theta_{M[P}\, X_{Q]N} - \theta_{N[P}\, X_{Q]M} \,, \\ [X_{MN},\, X_{P}] &= \theta_{P[N}\, X_{M]} - \tau_{[N}\, X_{M]P} \,, \\ [X_{M},\, X_{P}] &= \tau_{[P}\, X_{M]} - \frac{1}{2}\, \tilde{g}\, X_{MP} + g_{MPNQ}\, X^{NQ} \,, \\ \left[X_{M},\, X^{NP}\right] &= 3\, g_{P_1P_2P_3P_4}\, \epsilon^{NPP_1P_2P_3P_4S}\, X_{SM} - \frac{3}{14}\, \tau_M\, X^{NP} \,. \end{aligned} \tag{4.33}$$

Third example: Scherk-Schwarz gauging

As a second example let us consider the model originally studied by Cremmer, Scherk and Schwarz [54], which describes a generalized dimensional reduction of maximal $D = 5$ supergravity to $D = 4$, in which the Scherk-Schwarz twist is chosen in $\mathrm{E}_{6(6)}$, global symmetry group of the five dimensional theory. The resulting gauged supergravity [22] is defined by an embedding tensor transforming in the $\mathbf{78}_{+3}$ [37] with respect to the $\mathrm{E}_{6(6)} \times \mathrm{O}(1,1)$ subgroup of $\mathrm{E}_{7(7)}$. In the basis of the $\mathbf{56}$ in which the 28 electric vector fields are $A^\Lambda_\mu = \{A^u_\mu,\, A^0_\mu\}$, where A^u_μ, $u = 1, \ldots, 27$ are the dimensionally reduced five– dimensional vectors in the $\mathbf{27}_{-1}$ of $\mathrm{E}_{6(6)} \times \mathrm{O}(1,1)$ and A^0_μ is the Kaluza-Klein vector in the $\mathbf{1}_{-3}$ of the same group, the embedding tensor has just electric components θ^σ_Λ and the gauge generators read:

$$\begin{aligned} X_\Lambda &= \begin{cases} X_0 = \theta_{0,u}{}^v\, t_v{}^u \\ X_u = \theta_u{}^v\, t_v \end{cases} , \\ \theta_{0,u}{}^v &= \theta_u{}^v = M_u{}^v \in \mathrm{E}_{6(6)} \,. \end{aligned} \tag{4.34}$$

where $M_u{}^v$ is the twist matrix depending in general on 78 parameters, $t_u{}^v$ are the $\mathrm{E}_{6(6)}$ generators, and t_u are $\mathrm{E}_{7(7)}$ generators in the $\overline{\mathbf{27}}_{+2}$, according to the following branching of the $\mathrm{E}_{7(7)}$ generators with respect to $\mathrm{E}_{6(6)} \times \mathrm{O}(1,1)$:

$$\mathbf{133} \to \mathbf{78}_0 + \mathbf{1}_0 + \overline{\mathbf{27}}_{+2} + \mathbf{27}_{-2} \,. \tag{4.35}$$

In this case the relevant components of the gauge generators are:

$$\begin{aligned} X_{0u}{}^v &= -X_{u0}{}^v = -X_0{}^v{}_u = X_u{}^v{}_0 = -M_u{}^v \,, \\ X_{uvw} &= M_u{}^z\, d_{zvw} \,, \end{aligned} \tag{4.36}$$

where d_{uvw} denotes the three times symmetric invariant tensor of the $\mathbf{27}$ of $\mathrm{E}_{6(6)}$. To obtain eqs. (4.36) we have used the properties $(t_u{}^v)^w{}_z = -(t_u{}^v)_z{}^w = \delta^w_u \delta^v_z - (1/27)\, \delta^w_z \delta^v_u$, $(t_v)^w{}_0 = -(t_v)_0{}^w = \delta^w_v$ and $(t_u)_{vw} = d_{uvw}$. The gauge algebra has the following structure:

$$[X_0,\, X_u] = M_u{}^v\, X_v \,, \tag{4.37}$$

all other commutators vanishing.

If M is non-compact the corresponding theory depends effectively only on six parameters and the potential is of run-away type, namely there is no vacuum solution. If on the other hand M is compact, the theory has Minkowski vacua and depends effectively on four mass parameters, which determine the amount of residual supersymmetry on these solutions.

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A. The $\mathrm{E}_{7(7)}$ generators in the $\mathrm{GL}(7,\mathbb{R})$ basis

We give below the commutation relations among the $\mathrm{E}_{7(7)}$ generators in the basis (4.29):

$$
\begin{aligned}
\left[t_M{}^N,\, t_P{}^Q\right] &= \delta_P^N\, t_M{}^Q - \delta_M^Q\, t_P{}^N\,, \\
\left[t_M{}^N,\, t^{P_1P_2P_3}\right] &= -3\,\delta_M^{[P_1}\, t^{P_2P_3]N} + \frac{5}{7}\,\delta_M^N\, t^{P_1P_2P_3}\,, \\
\left[t_M{}^N,\, t_P\right] &= \delta_P^N\, t_M + \frac{3}{7}\,\delta_M^N\, t_P\,, \\
\left[t^{N_1N_2N_3},\, t^{P_1P_2P_3}\right] &= \epsilon^{N_1N_2N_3P_1P_2P_3Q}\, t_Q\,, \\
\left[t_M{}^N,\, t_{P_1P_2P_3}\right] &= 3\,\delta^N_{[P_1}\, t_{P_2P_3]M} - \frac{5}{7}\,\delta_M^N\, t_{P_1P_2P_3}\,, \\
\left[t_M{}^N,\, t^P\right] &= -\delta_M^P\, t^N - \frac{3}{7}\,\delta_M^N\, t^P\,, \\
\left[t_{N_1N_2N_3},\, t_{P_1P_2P_3}\right] &= \epsilon_{N_1N_2N_3P_1P_2P_3Q}\, t^Q\,, \\
\left[t^N,\, t_M\right] &= t_M{}^N + \frac{1}{7}\,\delta_M^N\, t\,, \\
\left[t^M,\, t^{N_1N_2N_3}\right] &= -\frac{1}{6}\,\epsilon^{MN_1N_2N_3P_1P_2P_3}\, t_{P_1P_2P_3}\,, \\
\left[t_M,\, t_{N_1N_2N_3}\right] &= -\frac{1}{6}\,\epsilon_{MN_1N_2N_3P_1P_2P_3}\, t^{P_1P_2P_3} \\
\left[t_{M_1M_2M_3},\, t^{N_1N_2N_3}\right] &= 18\,\delta^{[N_1N_2}_{[M_1M_2}\, t_{M_3]}{}^{N_3]} - \frac{24}{7}\,\delta^{N_1N_2N_3}_{M_1M_2M_3}\, t\,,
\end{aligned}
\tag{A.1}
$$

where $t \equiv t_M{}^M$.

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